My laptop has 6% power, so i'm writing this from TTY and it doesn't have russian keyboard
we're gonna be using:
$\langle \alpha f, x\rangle = \alpha \langle f, x\rangle = \langle f, \alpha x\rangle, \quad f \in \mathcal{L}^*$
Let $\{e_1, \dots, e_n\}$ eb a basis in $\mathcal{L} \implies \forall x \in \mathcal{L} \quad x^j e_j;$
$\langle e^j, x\rangle := x^j,\ j \in \overline{1, n}.$
We must prove: $\langle e^j, \alpha^1 x_1 + \alpha^2 x_2\rangle = \alpha_1 \langle e^j, x_1\rangle + \alpha_2 \langle e^j, x_2\rangle$
$x_1 = \alpha^1 x_1 + \alpha^2 x_2 = (\alpha^1 x_1 + \alpha^2 x_2)^j e_j = \alpha^1 x_1^j e_j + \alpha^1 x_2^j e_j = \left[(\alpha^1 x_1 + \alpha^2 x_2)^j - \alpha^1 x_1^j - \alpha^1 x_1^j\right] \cdot e_j = \theta \implies [] = 0 \implies$

$$(\alpha^1 x_1 + \alpha^2 x_2)^j = \alpha^1 x_1^j + \alpha^2 x_2^j$$

$(\alpha^1 x_2 + \alpha^2 x_2)^j = \langle e^j, \alpha^1 x_1 + \alpha^2 x_2\rangle;\ x_1^j = \langle e^j, x_1\rangle;\ x_2^j = \langle e^j, x_2\rangle \implies \langle e^j, \alpha^1 x_2 + \alpha^2 x_2\rangle = \alpha^1 \langle e^j, x_1\rangle + \alpha^1 \langle e^j, x_2\rangle$

Lemm 1.: $\{e^1, \dots, e^n\}$ forms a basis in $\mathcal{L}^*$
Prove:
fullness: $\langle f, x\rangle = \langle f, x^j e_j\rangle = x^j \langle f, e_j\rangle = \langle e^j, x\rangle \underbrace{\langle f, e_j\rangle}_{=\alpha}$
$\langle e^j, x\rangle \alpha = \langle \alpha e^j, x\rangle = \langle \langle f, e_j\rangle \cdot e^j, x\rangle \implies \forall x \in \mathcal{L} : f = \langle f, e_j\rangle e^j$ - we can
Linear independence: $\langle e^j, e_k\rangle = \left[e_k = 0e_1 + \dots + 1e_k + \dots + 0e_n = \delta_k^j e_j\right] = \delta_{jk}$
$\langle \alpha_j e^j, e_k\rangle = \langle \theta^*, e_k\rangle = 0 = \alpha_j \delta_k^j = \alpha_k, k \in \overline{1, k} \implies$ Linearly independant

Kinda important example: $\{P^n\}$ - set of polynoms with power $\le n, p^(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_n t^n$
basis: $\{1, t, t^2, \dots, t^n\}$ (to proove: fullness is trivial, linear independence proff goes like this: let $t = 0 \implies \alpha_0 = 0$, then differentiate n times and $\forall \alpha_i = 0$).
Taylor expansion of $p(t) \in \mathbb{P}^n = \sum_{k=0}^{n} = \frac{P^{(k)}(0)}{k!} t^k$ - which is same as basis expansion!
meanwhile, Lets try to find $\left(P^{(*)}\right)^*$
The operation: $\langle id \cdot |_{t=0}, P(t)\rangle := P(0), \left\langle \frac{d}{dt} \cdot \Big|_{t=0}, p(t)\right\rangle := P'(0), \dots, \left\langle \frac{d^n}{dt^n}\Big|_{t=0}, P(t)\right\rangle := P^{(n)}(0)$
so, basis in covector space $\left(P^{(*)}\right)^* : \left\{ id \cdot \Big|_{t=0}, \frac{d}{dt} \cdot \Big|_{t=0}, \frac{d^2}{dt^2}_{\dots}, \dots, \frac{d^n}{dt^n} \right\}_{\dots}$

Basis transition

we will mark **old basis** as $\{e_1, \dots, e_n\}$
and **new basis** is $\{e_{1'}, e_{2'}, \dots, e_{n'}\}$
and the "**new new" basis**: $\{e_{1''}, \dots, e_{n''}\}$
as you could guess, the "new new new" basis would be with three ticks

$e_{j'} = e_{j'}^i e_i \iff \begin{pmatrix} e_{1'} & \dots & e_{n'} \end{pmatrix} = \begin{pmatrix} e_1 & \dots & e_n \end{pmatrix} \begin{pmatrix} e_{1'}^1 & \dots & e_{n'}^1 \\ \vdots & \ddots & \vdots \\ e_{1'}^n & \dots & e_{n'}^n \end{pmatrix} \iff E' = E \cdot C$, where C is transition matrix, while $\det C \neq 0$. Let's prove that:
Let $\det C = 0. C = \left\| C_{1'}, \dots, C_{n'} \right\| \implies \exists \alpha^{i'} : \alpha^{1'} C_{1'} + \dots + \alpha^{n'} C_{n'} = 0$
$e_{i'} = c_{i'}^i e_i = \begin{pmatrix} e_1, \dots, e_n \end{pmatrix} \begin{pmatrix} c_{i'}^1 \\ \vdots \\ c_{i'}^n \end{pmatrix} = EC_{i'}, e_{i'} = EC_{i'}$
$\alpha^{1'} \cdot e_{1'}; E\alpha^{1'} \cdot C_{i'} = E0 = \theta$, that means that basis e' is linearly dependant $\implies \det C \neq 0$
$e_{i'} = c_{i'}^i e_i \iff E' = EC \implies \exists C^{-1} : E = E'C^{-1} \qquad C = (c_{i'}^i)_{n'}^n;\ C^{-1} = \left(c_i^{i'}\right)_n^{n'}$
This makes an easy way to denote whether the transition is from old ot new or from new to old basis

$$(e_1,\dots,e_n) = (e_{1'},\dots,e_{n'})\begin{pmatrix} b_1^{1'} & \dots & b_n^{1'} \\ \vdots & \ddots & \vdots \\ b_1^{n'} & \dots & b_n^{n'} \end{pmatrix} = E'B \implies \begin{cases} e_i = c_i^{i'} e_{i'} \\ e_{i'} = c_{i'}^{i} e_i \end{cases}$$ - **covariant law**

Basis coordinates transistion

$x = x^{j'} e_{j'};\ x = x^j e_j \implies ?x^{j'} = c_j^{j'} x^j?$

Proof: $e_j = c_j^{j'} e_{j'} \implies x^{j'} e_{j'} = x^j c_j^{j'} e_{j'} \iff \left[x^{j'} - c_j^{j'} x^j\right] e_{j'} = \theta \implies x^{j'} = c_j^{j'} x^j$ - **countervariant law**

Im matrix form:

$$\begin{cases} X'_e = C^{-1} X_e \\ E' = EC \end{cases}$$

Linear operator

Let there be $\mathcal{L}_1, \mathcal{L}_2$

Linear operator on $\mathcal{L}_1, \mathcal{L}_2$: $\mathcal{A} : \mathcal{L}_1 \to \mathcal{L}_2$ is a linear operator IF $\mathcal{A}(\alpha^1 x_1 + \alpha^2 x_2) = \alpha^1 \mathcal{A}(x_1) + \alpha^2 \mathcal{A}(x_2)$

Example: take $\mathbb{K}^{m\times n}, \mathcal{L}_1 = X \in \mathbb{K}^{n\times 1}, Y \in L_2 = \mathbb{K}^{m\times 1}$

$AX \in \mathcal{L}_2$, this is linear operator

kernel and image

ker $A := \{X \in \mathcal{L}_1 : AX_1 = \theta_2 \in \mathcal{L}_1\}$ - this is subset (or equal to) L_1 that gets you θ in $\mathcal{L}_2$ after applying A

im $A := \{Y \in \mathcal{L}_2 : \exists k \in L_1, y = A(X)\}$ - this is subset (or equal to) L_2 that you can get from $A(\mathcal{L}_1)$,

Lets prove that ker is subspace of $\mathcal{L}_1$, im is subspace of $\mathcal{L}_2$:

Let $X_1, X_2 \in$ ker $A : AX_1 = AX_2 = \theta_2 \implies A(\alpha^1 X_1 + \alpha^2 X_2) = \alpha^1 A(X_1) + \alpha^2 A(X_2) = \theta_2 \in$ ker A

and I CANT FUCKING SEE THE PROOF ON THE OTHER PART OF BLACKBOARD FUUUUCK: but the idea is (talke 2 vectors from image, take their linear combination and it appears to be also in that image